Onset of avalanches in granular media

Surajit Sen,[1] Normand Mousseau,[2,3] and Gregor Overney[2,4]

[1] *Physics Department, State University of New York, Buffalo, New York 14260*

[2] *Physics Department, Michigan State University, East Lansing, Michigan 48824-1116*

[3] *Theoretical Physics Department, University of Oxford, Oxford OX1 3NP, United Kingdom*

[4] *Chemistry Department, Michigan State University, East Lansing, Michigan 48824*

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Marginal stability and avalanches at angles $\Theta \geq \Theta_{\rm aval}$ are associated with granular media (GM) subject to special initial conditions. We study the Newtonian dynamics of random size-mismatched hard-core-like disks in a two-dimensional box which confirm claims that avalanches in GM are restricted to a few topmost boundary layers. We find that the velocity profile of the top layer grains, which obey $|v| \propto t^{\gamma}$, typically with $3.5 \leq \gamma$, signal the onset of an avalanche in GM. Our studies suggest that the dynamics of a single particle in a cosine potential in the presence of a linear field well describes the onset of motion of a top layer grain.

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Granular media (GM) such as dry sand, powders, glass beads, etc., possess characteristics of both the solid and the liquid phases. Granular systems are inherently inhomogeneous with highly anisotropic short ranged force networks which lead to their marginal stability and insure that they exhibit flow properties when inclined at angles equaling or exceeding some characteristic angle $\Theta_{\rm aval}$ subject to appropriate initial conditions, namely, the velocities of all the grains are zero at time $t = 0$ [1]. The issue of marginal stability has led to attempts to connect the dynamical properties of these systems with the slow dynamics in systems such as flux lattices, spin glasses, charge density wave systems, etc. [1, 2]. The physics of the onset of instabilities, i.e., of avalanches, in GM concerns a complicated dynamical process far from equilibrium that not only renders these systems interesting from a dynamics standpoint but also for important practical reasons such as predicting geological processes (such as mass wasting via mudslides and landslides, sand and snow avalanches, plate tectonics, and earthquakes) as well [3]. The issue of the onset of avalanches in GM is the central theme of this Brief Report.

Experiments and some simulations on spherical grains demonstrate that the hard-core-like interactions which characterize GM [1, 2, 4] are responsible for the nonclose packed structure of these systems in their metastable states and hence for their marginal or reduced stability [1, 2, 4, 5]. More precisely, experiments lead to the following observations: (i) When dry GM are inclined at a slope that exceeds a characteristic angle, the boundary layers (typically < 6) begin to flow [1]. (ii) Given the strong gravitational field compared to the nearly hard-core-like interaction between the grains, thermal energy is vanishingly small, and hence the physics of GM can be studied at temperature $T = 0$. (iii) The motion of the grains, especially those in the boundary layers, is "sticky" in nature (usually explained via the notion of friction). (iv) Recent studies of Jaeger *et al.* [6] demonstrate that for a system of glass beads of mean radius 0.27 mm with about 14% size mismatch, avalanche occurs at $\Theta_{\rm aval} = \Theta_r + \delta$, where $\Theta_r = 26°$ is the angle between the horizontal and the free surface of the GM *after* an avalanche has restored the pile to a metastable equilibrium slope and $\delta \approx 2.6°$. (v) Recent studies have demonstrated that the sound propagation in GM is primarily along the vertical direction (due to gravitational force) with sound velocity $\nu \propto P^{1/6}$, P being the pressure, while the horizontal sound velocity is nearly zero, which bear testimony to the approximately hard-core-like nature of the intergranular interactions [1]. As we shall show, our studies on the onset of avalanches in a two-dimensional (2D) system with hard-core-like intergrain potential lead to some results and to some observations that are consistent with (i)–(v) enumerated above.

Our study can be summarized as follows: (i) We numerically solve the Newtonian equations for a short-ranged interacting system of hard-core-like disks in a gravitational field in a *finite* box tilted at an angle Θ. (ii) Our calculations reveal that boundary layer flow [(i) above] as well as the observed $\Theta_{\rm aval}$'s are readily understandable without any special assumptions on the time evolution process. The parameters of the potential are chosen from available experimental information [6] and readily reveal that a $T = 0$ study is adequate [see (ii) above]. (iii) Although our calculations do not invoke friction, they nevertheless demonstrate (for the chosen two-body potential) that the motion of a typical grain is "sticky" in nature [see (iii) in the preceding paragraph]. (iv) We find that the onset of an avalanche in GM is typically characterized by a dramatic rise in the average absolute velocity profile (AAVP) of the top layer grains, a feature that appears not to have been well explored before. We suggest that such dramatic behavior may, in principle, be exploited to predict the onset of avalanches in model systems and eventually even in geological contexts [7]. (v) We show that the velocity profile of a single top layer grain at the onset of an avalanche can be explained in terms of a 1D one-body model—that of a particle initially at rest which later traverses out of a corrugation potential upon being driven by a linear gravitational field. While this explanation is not meant to undermine the collective origins of the onset process, it does suggest that at the onset, a set of top layer grains behave as independent particles on some corrugation modulated

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surface. The details of our study are given below.

We assume a truncated Slater-Buckingham pair potential [8] between the hard-core-like disks. In addition, the disks experience a gravitational field. The Hamiltonian $H = \sum_{i=1}^{N} \left(\frac{p_i^2}{2m} + V_i\right) + \frac{1}{2}\sum_{i>j} V_{ij}$, where

$$V_{ij} = \frac{\epsilon}{1-6/\alpha}\left\{\frac{6}{\alpha}\exp\left[\alpha\left(1-\frac{r_{ij}}{\sigma}\right)\right] - \left(\frac{\sigma}{r_{ij}}\right)^6\right\} + \epsilon, \tag{1}$$

$$V_i = mgh_i, \tag{2}$$

and where we choose r_{ij} such that $dV_{ij}/dr_{ij} \leq 0$ and the softness or hardness of the disks is tunable via α. In the present study (with a 2D system), we choose the following parameters in order to compare our 2D study with the available experiments to the extent that is feasible. Thus we assume $\epsilon = 5\times10^{-3}$ kg(m/s)^2, $\alpha = 55$, and $\sigma = r_i + r_j$, $r_i = 10^{-3}$ m, $m = 10^{-4}$ kg. For the rotational kinetic energy (of each grain) to be comparable with the translational kinetic energy, an angular velocity of $\omega \approx 2000$ s^{-1} is needed [4, 9]. This is unusually high and thus justifies ignoring rotational effects in this first study. The time integration of the equations of motion are carried out via a fourth-order Gear predictor-corrector algorithm [10]. The integration time step $\Delta t = 5 \times 10^{-7}$ s is intentionally chosen to be small enough to consider effects of the short range intergranular interactions accurately for this hard-disk-like GM. The time integral is carried out up to $t_{\max} = 10^{-1}$ s, which turns out to be adequate for studying the commencement of the avalanche in our 2D system.

The coupled equations of motion are $\sum_{i=1}^{N} m\frac{d\mathbf{v}_i}{dt} = -\sum_{i=1}^{N} \nabla(V_{ij}+V_i)$, where N is the total number of grains each of fixed mass m. In the first and simplest study, we start with a lattice with no size mismatch between the grains and with every disk (or grain) at rest and arranged in a close packed triangular lattice in a box with the base oriented parallel to the horizon. Given that the triangular lattice provides the most stable structure in $2D$, this configuration yields an excellent and highly stable equilibrium structure to start from. For uniform hard-core disks, geometry dictates that $\Theta_{\text{aval}} = 30°$ while $\Theta_{\text{aval}} < 30°$ for a soft disk case. We have also studied avalanches in systems with small size mismatch (e.g., $\approx 1\%$ between the grains) as we shall describe below. In these studies the system is relaxed to the lowest energy, which in general yields a distorted triangular lattice, which in turn serves as the equilibrium structure at $\Theta = 0$ and the starting point of our studies. As one would expect, the introduction of mismatch renders the system dynamically more responsive with respect to the gravitational field than when compared to the pristine one-component system. The walls of the box enclosing the lattice are kept perfectly elastic except when relaxing the system as will be discussed below. Studies with larger systems with mismatches as large as $\approx 10\%$ and randomly packed have also been carried out [4, 11]. Preliminary results indicate that the nature of the onset of an avalanche remains essentially invariant for these larger mismatches when random packing is enforced [4].

Our calculations were performed on a Convex C240 computer. Starting from the relaxed lattice at $\Theta = 0$, the box is "adiabatically tilted" by some angle $\Delta\Theta$ (i.e., the grain coordinates are transformed via the rotation angle $\Delta\Theta$) and allowed to come to the lowest energy. Also, every grain is brought to rest (recall $T = 0$) after the box is tilted by draining the vibrational kinetic energy of the grains via highly inelastic collisions between the grains and the walls, an approach which works efficiently for "small" systems. This "adiabatic tilting" is not only important to render the numerical study of the model system physically realistic, but also for the sake of carrying out the next important step, i.e., study the onset of an avalanche in a meaningful way. This dissipation is removed after the system has relaxed completely. It turns out that the "adiabatic tilting" can be accomplished in steps of a few degrees at low tilt angles and at increments of 0.5° or less at angles near (what becomes) Θ_{aval}, i.e., the angle in which the avalanche commences as signalled by *collective motion of the top layer grains along the incline.* This collective motion can be well characterized by a simple quantity, the AAVP parameter (checked for self-consistency over several runs) defined as $|v_x| \equiv |\sum_{N_1} v_i^x/N_1|$, where v_i^x denotes the velocity along the incline of each of the N_1 grains in the top layer, which shows a dramatic power law growth in time at the onset of an avalanche and eventually grows linear in time [Fig. 1(a)]. A power law growth is also obtained for the corresponding position profile parameter defined in the same way as above with x_i instead of v_i^x. The numerical accuracy of Θ_{aval} is, in practice, sensitive to the details of the "adiabatic tilting" process in marginally stable systems such as those which are far from equilibrium, and hence Θ_{aval} is inevitably accuracy limited in a numerical study. We have therefore paid careful attention to the manner in which Θ_{aval} was obtained and every calculation has been repeated several times for reliable determination of Θ_{aval}. The top layer velocity and the top layer position profiles perpendicular to the surface are, as one would expect, dominated by fluctuations. No clear-cut signature appears to be associated with the onset of an avalanche in the direction perpendicular to the incline in our 2D model. However, there are interesting effects that emerge in the layers immediately below the top layer preceding an avalanche for the close packed systems studied. These effects are described in Figs. 1 and 2. Figure 1 clearly shows that the onset of an avalanche occurs at $t \approx 0.04$ s in both the bimodal and the random size mismatch systems. Figure 2 reveals that the onset of an avalanche is *significantly influenced by the boundary* (observe the fifth grain from the right boundary and its vicinity in Fig. 2 at $t = 0.01$ s). Our studies suggest that typically the boundary helps nucleate defects in deeper layers which eventually distort the "corrugated surface" in the top most layer. An important issue concerning the boundary is that our boundary conditions are qualitatively consistent with those used in the experiments of Jaeger *et al.* [1] and that a different boundary condition could lead to different avalanche angles. However, as we shall show below, for the chosen boundary conditions, the results appear to be essentially independent of the

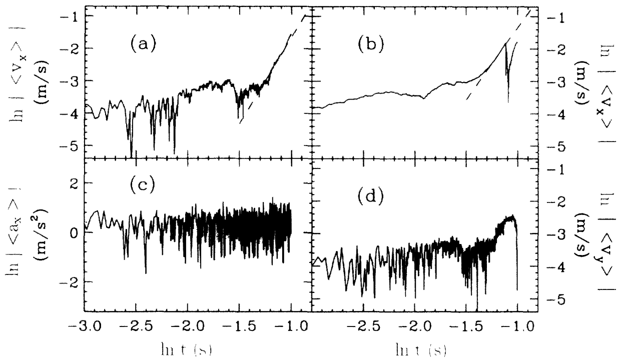

FIG. 1. (a) $\ln|v_x|$ vs $\ln t$ for bimodal mismatch case (slope=5.36 at avalanche); (b) $\ln|v_x|$ vs $\ln t$ for random mismatch case (slope=4.46 at avalanche). The dip in (b) near $\ln t \approx 1$ originates in reflection from the right wall; (c) and (d) Log of average acceleration and average velocity parallel and perpendicular to the plane, respectively, in the bimodal mismatch system.

system size.

The main focus of our study is to achieve some understanding of the experimentally observed properties (i)–(v) enumerated in the beginning of this Brief Report. We have studied two specific system sizes. The first study involved a 212 grain system with eight layers [11]. This study established that the avalanche is a boundary layer process involving the first couple of layers at the early stages of the avalanche and involves top layer flow and will be discussed elsewhere [4,11]. We then simplified and enlarged our system by studying a 418 grain system and a 834 grain system with only four layers and a corrugation potential which replaced the fifth and lower layers via a one-body corrugation potential chosen to approximately describe the close packed nature of the granular system in layers deep below the surface. Given that our focus lies on the problem of onset of an avalanche, the introduction of a corrugation potential to mimick a layer sufficiently below the interface makes little or no discernible difference in the processes that lead to the commencement of the avalanche.

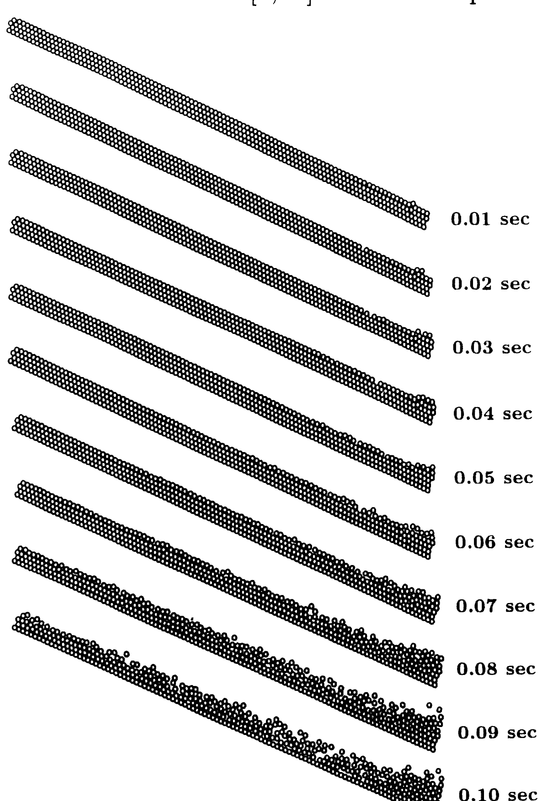

FIG. 2. Onset of an avalanche in the box with corrugated base as a function of real time for the model system. Observe that the avalanche is nucleated near the right boundary.

We find that the average velocity profile of the top layer grains exhibits the characteristic behavior given by $|v_x| \propto t^\gamma$, where typically we find $4 \leq \gamma \leq 6$ for the chosen potential. The corresponding average position profile, not surprisingly, is consistent with $|x| \propto t^{\gamma+1}$. Our numerical studies reveal a $\Theta_{\mathbf{aval}}$ with an accuracy of $\approx \pm 1.0°$. The role of size mismatch at the onset of an avalanche is not well understood. We have carried out our calculations for three distinct systems with mismatches $\mu = 2(r_i - r_j)/(r_i + r_j)$, being zero in one, and 0.01 in the other two, where r_i defines the radius of any disk i. We consider both bimodal (i.e., system with only two different disk sizes, 0.995×10^{-3} m and 1.005×10^{-3} m) and linear random (i.e., continuous distribution of mismatches with disk radii distributed between 0.995×10^{-3} m and 1.005×10^{-3} m) size mismatches in our study. It turns out, as mentioned earlier, that the experimental studies of Jaeger *et al.* [6] involved much larger size mismatches, ≈ 0.14 in 3D, which was intrinsically more stable than a 2D GM. We find that the velocity profiles are insensitive to the mismatch parameter for sufficiently small mismatches (such as $\mu \approx 0.01$). In addition, as we shall see below, we find that $\Theta_{\mathbf{aval}}$ is at best only weakly dependent upon the nature of the mismatch for sufficiently small mismatches within the accuracy of our calculations.

Simple geometry reveals that in a hard disk system with zero mismatch, $\Theta_{\mathbf{aval}} = 30°$. The value of $\Theta_{\mathbf{aval}}$ for our hard-disk-like system is $30° - \Delta\Theta$, typically with $\Delta\Theta \leq 4°$. The departure from the 30° value is a measure of the softness of the potential and is non-trivial to track down accurately numerically due to limitations involved in the "adiabatic tilting" calculations mentioned earlier. In the bimodal system with $\mu = 0.01$ and 418 grains we find that the first avalanche occurs at $\Theta_{\mathbf{aval}} = 26.7°$ [see Fig. 1(a)]. For a system with random linear size mismatch with $\mu = 0.01$ and 418 grains we find that an avalanche occurs at $\Theta_{\mathbf{aval}} = 27.7°$ [see Fig. 1(b)]. It turns out that constraints on computational power, which constrains the accuracy of the "adiabatic tilting" process, possibly lead us to underestimate $\Theta_{\mathbf{aval}}$ for the two systems with a maximum estimated inaccuracy of $\pm 1.0°$ in each case. This latter system is therefore more stable than the former as one would intuitively expect. The numbers are reasonable compared to the experiments of Jaeger *et al.* [6] which reported $\Theta_{\mathbf{aval}} \approx 29°$ for a 3D system. We have also carried out our calculations for an 834 grain system for which the bimodal mismatch system yields $\Theta_{\mathbf{aval}} = 27.8°$ and the linear random mismatch system yields 28.6°. For any given small mismatch the avalanche (characterized by collective motion of top layer grains) commenced at about the same time, between 0.04 and 0.05 s (see Fig. 1) a finding that remained invariant over a large number of independent calculations. We found that the characteristic time at which the avalanche commences is approximately independent of the size mismatch for small mismatches and depends only upon the hard-disk-like nature of the potential. Clearly, one would expect this characteristic time to depend on parameters such as mismatch and the softness of the potential and the strength of the gravitational field. The details of this dependence requires more extensive analysis of the parameter space for α, σ, g, and μ and will be addressed elsewhere [11].

Figures 1(a) and 1(b) present the average velocity profile parameters, $|v_x|$, for the bimodal and random size mismatch cases, and Figs. 1(c) and 1(d) present the average acceleration (velocity) profile parameter along $x(y)$ direction for the top layer for the system with bimodal size mismatch. The averages are taken over the top layer grains and calculated by considering 54 grains distributed about the center of the top layer. This procedure insures that the boundary effects do not contaminate the velocity profile parameter. Our studies reveal that $|v_x| \propto t^\gamma$, where $\gamma = 5.36$ and 4.46 for the bimodal and linear random mismatch systems, respectively, in Figs. 1(a) and 1(b). The corresponding γ's for the 834 grain system yielded 4.55 and 4.80, respectively. We have, in addition, checked our results for a case with $V_i = 10mgh$ in Eq. (2). Interestingly, the results remained invariant except for a reduction in $\Theta_{\rm aval}$ which was 21.0° and 22.2° for the bimodal and random mismatch systems, respectively. The parameter γ was found to be 4.23 and 4.57 for these two cases, respectively.

The unusual behavior of the velocity profile parameter $|v_x| \propto t^\gamma$ where $4 \leq \gamma$ at the onset of an avalanche can be understood via a simple 1D one body model with appropriate choice of values of the parameters described as follows. Consider a particle being acted upon by a linear potential field in a corrugation potential defined by the following Hamiltonian (see the inset in Fig. 3): $H = p^2/2M + A\cos(kx) + Kx$, which gives $M\frac{d^2x}{dt^2} = Ak\,\sin(kx) - K$, where $2A$ is the height difference between the crest and the trough of the corrugation potential, k is the corrugation period, and K is the coupling to the linear field (in this case, the gravitational field). Both A and K depend on the direction of the gravitational field with respect to the surface of the top layer of the GM. Choosing parameters in agreement with the physical model $M = 10^{-4}$ kg, $A = 2.1\times10^{-7}$ J, $k = \pi\times10^3$ m^{-1}, and $K \approx -5.05\times10^{-4}$ N and choosing an initial velocity $v(0) = 0$ and initial position $x(0) = 1/1000$ which guarantee that the particle is at rest at the trough of the cosine potential at $t = 0$ and integrating the equation of motion in time, we obtain the velocity profile shown in Fig. 3. Interestingly, our parameters give a velocity profile with $\gamma = 4.58$, in excellent agreement with the numbers obtained from the extensive numerical analysis described here. The strength of the power law growth of the velocity is sensitive to the values of the parameters chosen and hence stronger growth laws are also conceivable. This assertion can be verified via a short time expansion study of $v(t)$. This agreement illustrates the power of this one dimensional simple model and helps understand how γ might change when a significantly stiffer potential must be considered, i.e., with a larger A or a smaller k or both. Our model reproduces the velocity profile parameter, a many-body quantity, via a one-body picture at the *onset* of an avalanche, thus suggesting that the problem of onset is largely dictated by the local geometry of the granular medium.

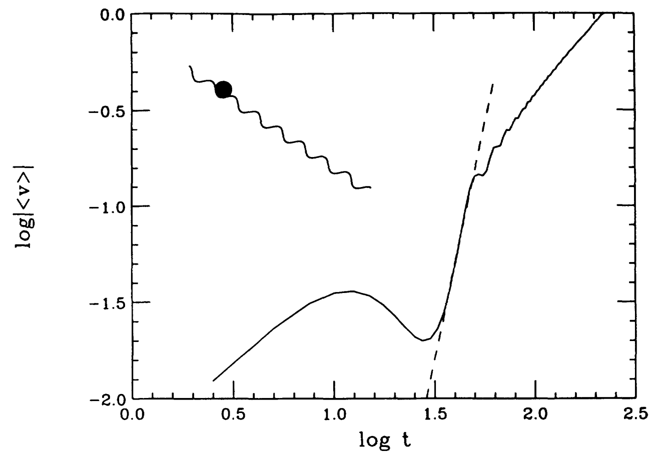

FIG. 3. The velocity profile of a particle acted upon by a linear field in a corrugation potential (inset). Parameters are chosen to describe the model many body system and yield a slope ≈ 4.58, in remarkable agreement with many body calculation results.

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